Appendix to: Trade and growth with heterogeneous firms revisited

1 Setup of the Model

Time is continuous and the world economy is composed by two symmetric regions, each endowed with L workers who inelastically supply one unit of labour at every moment in time. Labour can be devoted to the production of final consumption goods or intermediate knowledge goods. Different varieties of the consumption good are produced monopolisitcally by manufacturing firms with heterogeneous productivity. The innovation sector produces knowledge enabling the emergence of new consumption varieties over time as in the standard model of endogenous growth with expanding product varieties.

A.1 Consumers

Consumers have to make two choices. First, they choose how much to consume and save at each moment in time, i.e. they decide their optimal expenditure level $E(t)$. Then, they establish how to split their consumption among the different varieties of final goods available at each t.

A.1.1 Dynamic decision

Welfare at t is defined as the present discounted value of future real consumption of the final good composite:

$$U(t) = \int_t^{\infty} e^{-\rho(s-t)} \ln\left[D(s)\right] \mathrm{d}s \tag{A.1}$$

where $\rho > 0$ is the rate of pure time preference, $D(t) = E(t)/P(t)$ is real consumption of the final good composite and $P(t)$ is the aggregate price index at t.

At every moment t, consumers maximize (A.1) subject to the budget constraint $Y(t) = E(t) + S(t)$ where $S(t)$ are savings and $Y(t)$ is current income and is composed by earnings from labour and profits made by domestic firms, i.e. $Y(t) = w(t)L + \Pi(t)$. Wages are taken as the numeraire ($w = 1$). Let me define $A(t)$ as accumulated savings at t, i.e. $\dot{A}(t) = S(t)$. Then assuming that financial market is perfect implies that revenues from owning firms gives households the same return than investing their accumulated wealth at rate $r(t)$, i.e. $\Pi(t) = r(t)A(t)$. We can therefore write the restriction of the dynamic problem as $\dot{A}(t) = L + r(t)A(t) - E(t)$. The resulting Hamiltonian to the consumers' problem is:

$$H(E, A, \lambda, t) = e^{-\rho t} \ln[E(t)/P(t)] + \lambda(t)[L + r(t)A(t) - E(t)]$$

where λ(t) is the multiplier. In this dynamic maximization problem both the price level $P(t)$ and the rate of return on savings $r(t)$ are exogenous to the consumer so her control variable is only the expenditure level $E(t)$. First order conditions for the consumers' problem are a transversality condition and:

$$
\begin{aligned}
- \quad & \frac{\partial H(t,\lambda)}{\partial E(t)} = 0 \Rightarrow e^{-\rho t}\frac{1}{E(t)} = \lambda(t) \Rightarrow \dot{\lambda}(t) = -\rho e^{-\rho t}\frac{1}{E(t)} - e^{-\rho t}\frac{\dot{E}(t)}{E(t)^2} \\
& \Rightarrow \frac{-\dot{\lambda}(t)}{\lambda(t)} = \rho + \frac{\dot{E}(t)}{E(t)} \\
- \quad & \frac{\partial H(t,\lambda)}{\partial A(t)} = -\dot{\lambda}(t) \Rightarrow r(t)\lambda(t) = -\dot{\lambda}(t) \Rightarrow \frac{-\dot{\lambda}(t)}{\lambda(t)} = r(t)
\end{aligned}
$$

By dividing the results of these conditions we get the following Euler equation

$$
\frac{\dot{E}(t)}{E(t)} = r(t) - \rho \tag{A.2}
$$

which gives the optimal path for expenditure in our model.

A.1.2 Static decision

Consumers have CES preferences for final products reflected in their demand for the final consumption good bundle

$$
D(t) = \left[\int_{\theta\in\Theta(t)} d(\theta,t)^{1-1/\sigma}\mathrm{d}\theta\right]^{1/(1-1/\sigma)} \tag{A.3}
$$

where $d(\theta,t)$ represents the demand for variety θ, $\Theta(t)$ represents the mass of available varieties in the final-good market of this economy (both produced domestically and imported) at time t and $\sigma > 1$ is the constant elasticity of substitution between any two varieties. With Dixit-Stiglitz competition in the market of final goods, the perfect price index in the market with monopolistic competition can be written as

$$
P(t) = \left[\int_{\theta\in\Theta(t)} p(\theta,t)^{1-\sigma}\mathrm{d}\theta\right]^{1/(1-\sigma)} \tag{A.4}
$$

where $p(\theta,t)$ is the price of variety θ at time t.

Using these expressions we can find the optimal consumption for each variety. Consumers maximize (A.3) constrained by the expenditure level E determined in their dynamic problem. Using an increasing transformation of the objective function we can write the Lagrangian of this static problem as $\mathcal{L}_t = D(t)^{(\sigma-1)/\sigma} + \epsilon[E(t) - \int_\Theta p(\theta,t)d(\theta,t)\mathrm{d}\theta]$, where ϵ is the Lagrangian multiplier and first order conditions are:

$$
\begin{aligned}
\frac{\partial \mathcal{L}_t}{\partial d(\theta,t)} &= \frac{\sigma-1}{\sigma} d(\theta,t)^{\frac{\sigma-1}{\sigma}-1} - \epsilon(t)p(\theta,t) = 0 \quad \forall\theta, t \\
\frac{\partial \mathcal{L}_t}{\partial \epsilon(t)} &= E(t) - \int_{\theta\in\Theta(t)} p(\theta,t)d(\theta,t)\mathrm{d}\theta = 0
\end{aligned}
$$

For a given variety θ_1 we can use the first condition to write:

$$
\frac{\sigma-1}{\sigma} d(\theta_1,t)^{\frac{-1}{\sigma}} = \epsilon p(\theta_1,t)
$$

Replicating for another variety θ_2 and dividing both results we obtain relative demand: $\left[\frac{d(\theta_1,t)}{d(\theta_2,t)}\right]^{\frac{-1}{\sigma}} = \frac{p(\theta_1,t)}{p(\theta_2,t)}$. This expression states that the lower the relative price of variety

θ_1 the greater its relative demand. Since the demand for all varieties enter aggregate demand symmetrically we can use the previous result for any variety θ and rearrange it as $d(\theta,t) = \left[\frac{p(\theta,t)}{p(\theta^*,t)}\right]^{-\sigma} d(\theta^*,t)$ where θ^* is any variety different from θ. Then we can insert this result into the second condition obtaining:

$$E(t) = \int_{\theta\in\Theta(t)} p(\theta,t)d(\theta,t)\mathrm{d}\theta = d(\theta^*,t)p(\theta^*,t)^{\sigma} \underbrace{\int_{\theta\in\Theta(t)} p(\theta,t)^{1-\sigma}\mathrm{d}\theta}_{P(t)^{1-\sigma}}$$

From which we can generalize an expression for the demand of any variety θ:

$$d(\theta,t) = \frac{E(t)}{P(t)}\left[\frac{p(\theta,t)}{P(t)}\right]^{-\sigma} \tag{A.5}$$

Aggregate expenditure is given by:

$$E(t) = \int_{\theta\in\Theta(t)} e(\theta,t)\mathrm{d}\theta \quad \text{with} \quad e(\theta,t) = E(t)\left[\frac{p(\theta,t)}{P(t)}\right]^{1-\sigma} \tag{A.6}$$

A.2 Final good producers

Potential entrants to the final good sector must incur in a sunk cost F_I to buy a blueprint and discover their marginal cost of production a which is drawn from a distribution $g(a)$ with cumulative distribution $G(a)$ and $0 \leq a \leq a_0$. There are no costs for a firm to differentiate the variety they produce from those of other firms. This, together with the fact that all varieties enter the demand function symmetrically (by A.3) provides incentives for every firm to produce a different variety of the final good. Then, we can refer to firms using their unique variety index θ or marginal cost a instinctively. A producing firm enjoys monopolistic rights on the variety they produce until they are hit with a negative shock that forces them out of production. The negative shock happens with probability $\delta > 0$. Once they know how productive they will be if producing, firms face no other source of uncertainty and are able to estimate their future stream of revenues. Final producers must sink two extra costs: one for selling products in the domestic market (F_D) and another one for doing so abroad (F_X). Each of these costs can be expressed as $F_i = \kappa_i P_K$ for $i = I, D, X$ where κ_i represents the cost in terms of knowledge. Besides the sunk costs, exporters also face an iceberg cost of $\tau \geq 1$ to sell abroad.

Let me define m as the marginal selling cost of a final producer with marginal cost a, then $m = a$ in the domestic market and $m = \tau a$ in the foreign market. Using the demand function for a variety in (A.5) we can determine revenues for the firm with marginal cost m in a given market:

$$r(m,t) = p(m,t)d(m,t) = E(t)\left[\frac{p(m,t)}{P(t)}\right]^{1-\sigma} \tag{A.7}$$

Notice that, by (A.6), revenues a firm makes in a given market equals expenditure of consumers from that market in the good the firm is selling.

Operating profits (i.e. profits after sunk costs) of a firm with marginal cost m in a certain market are:

$$\pi(m,t) = r(m,t) - wl(m,t)$$

where $l(m,t) = d(m,t)m$ is the labour requirement of the firm and $w = 1$.

We can find the optimal pricing rule for a firm by using its residual demand given by (A.5) which can be rewritten as $p(m,t) = [A(t)/d(m)]^{1/\sigma}$, with $A(t) = E(t)P(t)^{\sigma-1}$ representing aggregate market conditions. This allows us to write:

$$\pi(m,t) = d(m,t)^{1-1/\sigma}A(t)^{1/\sigma} - d(m,t)m$$

and the first order condition for the optimum is

$$\frac{\partial \pi(m,t)}{\partial d(m,t)} = \frac{\sigma-1}{\sigma}\underbrace{d(m,t)^{\frac{-1}{\sigma}}A(t)^{\frac{1}{\sigma}}}_{p(m,t)} - m = 0$$

which implies that each firm follows the next pricing rule:

$$p(m) = \frac{m}{1-1/\sigma} \tag{A.8}$$

The firm with marginal cost m charges a constant mark-up over its constant marginal cost that depends solely on the elasticity of substitution σ.

A firm with marginal cost m has a market share of $s(m,t) = r(m,t)/R(t)$ where $R(t) = P(t)D(t)$ are total revenues in the economy. The fact that both economies in our model are symmetric implies that revenues that domestic firms make in the foreign market equal revenues that foreign firms make in the domestic market. By this property total expenditure in a given market must equal total revenues by firms in that market, i.e. $R(t) = E(t)$. We can then write market share as follows:

$$\begin{aligned}
s(m,t) &= \frac{r(m,t)}{E(t)} \overbrace{=}^{(A.7)} \left[\frac{p(m)}{P(t)}\right]^{1-\sigma} \\
&\underbrace{=}_{(A.8)+(A.4)} \left[\frac{m}{1-1/\sigma}\right]^{1-\sigma}\left[\int_{\theta\in\Theta_t}\left(\frac{m(\theta)}{1-1/\sigma}\right)^{1-\sigma} \mathrm{d}\theta\right]^{-1} \\
&= m^{1-\sigma}\left[\int_a m(a)^{1-\sigma}n(t)\mathrm{d}G(a)\right]^{-1} = \frac{1}{n(t)}\left[\frac{m}{\bar{m}(t)}\right]^{1-\sigma}
\end{aligned} \tag{A.9}$$

where

$$\bar{m}(t)^{1-\sigma} = \int_0^{a_D} a^{1-\sigma}\mathrm{d}G(a|a_D) + \phi\int_0^{a_X} a^{1-\sigma}\mathrm{d}G(a|a_D) \tag{A.10}$$

and $0 \leq \phi = \tau^{1-\sigma} \leq 1$. Here $\bar{m}$ is a benchmark selling cost and ϕ is a measure of variable trade costs (when $\tau \to 1 \Rightarrow \phi \to 1$ and when $\tau \to \infty \Rightarrow \phi \to 0$). The pricing rule in (A.8) implies each final producer has a mark-up over its sales of $1/\sigma$ so the operating profits a firm with selling cost m makes is

$$\pi(m,t) = \frac{s(m,t)E(t)}{\sigma} \tag{A.11}$$

Finally, aggregate operating profits are $\Pi(t) = E(t)/\sigma$.

A.3 Knowledge sector

Production of knowledge follows:

$$Q_K(t) = \frac{L_K(t)}{c(t)} \tag{A.12}$$

where L_K is the amount of labour devoted to the development of knowledge and $c(t) = c(w, \overrightarrow{a}, n(t))$ is the marginal cost of this activity. The marginal cost of innovating is determined by labour costs $w = 1$, the distribution of final good producer's costs $\overrightarrow{a}$ and the amount of existing blueprints in the economy n. The only property imposed to $c(.)$ is to be homogeneous of degree minus one in n which captures knowledge spillovers within the economy. Perfect competition in the market of knowledge sets its price equal to its marginal cost, so $P_K = c$ at every moment in time. Then, the properties of $c(.)$ allow to express the price of knowledge in intensive terms $nP_K = p_K = p_K(\overrightarrow{a})$ which depends solely on the distribution of final good producers.

New varieties enter the economy following $\dot{n}(t) = Q_K(t)/\bar{\kappa}(t) - \delta n(t)$, where $\bar{\kappa}(t)$ is the average amount of knowledge demanded by final good firms at t and can be defined as

$$\bar{\kappa} = \frac{\kappa_D G(a_D) + \kappa_X G(a_X) + \kappa_I}{G(a_D)} \tag{A.13}$$

Defining as $g(t) = \frac{\dot{n}(t)}{n(t)}$ the growth rate of the mass of varieties, we reach

$$g(t) = \frac{L_K(t)}{p_K \bar{\kappa}(t)} - \delta \tag{A.14}$$

2 Static equilibrium

At any moment of time t, the mass of varieties n is determined by history, the value of winners is set by the expected flow of future operating profits, consumers establish their optimal consumption level E and an optimal allocation of labour is decided (L_K is known). Free-entry in the production of final goods imposes that at every moment in time there is a firm with marginal cost $a_D(t)$ for which operating profits exactly offset the sunk cost of selling domestically. A similar marginal cost threshold $a_X(t)$ exists for selling abroad. The cut-off conditions are:

$$\frac{s(a_D)E}{\sigma\gamma} = P_K \kappa_D \quad ; \quad \frac{\phi s(a_X)E}{\sigma\gamma} = P_K \kappa_X \tag{A.15}$$

where γ is the discount rate at a given t. Free-entry also mandates that the ex-ante value of engaging final good production should equal the average sunk costs of the activity. Let us define the average fixed cost paid by producing firms at t as $\bar{F} = P_K \bar{\kappa}$ with $\bar{\kappa} = [\kappa_D G(a_D) + \kappa_X G(a_X) + \kappa_I]/G(a_D)$. Then we can write the following free entry condition:

$$\frac{E}{n\sigma\gamma} = P_K \bar{\kappa} \tag{A.16}$$

Conditions in (A.15) and (A.16) give cut-off values a_D and a_X. Once we know the distribution of producing and exporting firms, i.e. $G(a|a_D)$ and $G(a|a_X)$ respectively, we can find the selling cost of the benchmark firm $\bar{m}$ as well as the average sunk cost in units of knowledge paid by final good producers $\bar{\kappa}$. To reach the expressions in

BRN we need to assume a Pareto distribution for firm's productivity, which amounts to establish $G(a) = (a/a_0)^k$ and setting $a_0 = 1$. We also need to define $T = \kappa_X/\kappa_D$, $\beta = k/(\sigma - 1) > 1$ and $\Omega = \phi^\beta T^{(1-\beta)}$. Parameter Ω is key for our analysis as it bundles together the two types of trade costs (sunk T and variable ϕ) into one measure of trade openness. When $\Omega \to 0$ the economy is close to autarky while when $\Omega \to 1$ full integration is achieved.

Inserting $G(a) = a^k$ into (A.10) yields

$$\bar{m}^{1-\sigma} = \frac{\beta}{\beta - 1} a_X^{1-\sigma} \left[\left(\frac{a_X}{a_D} \right)^{\sigma - 1} + \phi \left(\frac{a_X}{a_D} \right)^k \right] \tag{A.17}$$

while introducing that specification for $G(.)$ into (A.13) gives

$$\bar{\kappa} = \kappa_D + \kappa_X \left(\frac{a_X}{a_D} \right)^k + \kappa_I \frac{1}{a_D^k} \tag{A.18}$$

Dividing both expressions in (A.15) we get

$$\frac{a_X}{a_D} = \left(\frac{T}{\phi} \right)^{1/(1-\sigma)} \tag{A.19}$$

Plugging the second expression in (A.15) into (A.16) we get

$$\left(\frac{\bar{m}}{a_X} \right)^{1-\sigma} = \frac{\bar{\kappa}\phi}{\kappa_X} \tag{A.20}$$

in which we can insert (A.17)-(A.20) to get a parametric solution for a_D

$$a_D = \left[\frac{\kappa_I(\beta - 1)}{\kappa_D(1 + \Omega)} \right]^{1/\alpha} \tag{A.21}$$

Then, we can take this result back to (A.19) and obtain

$$a_X = \left[\Omega \frac{\kappa_I(\beta - 1)}{\kappa_X(1 + \Omega)} \right]^{1/\alpha} \tag{A.22}$$

We can take these cut-off values back to (A.17) and (A.18) to obtain

$$\bar{m}^{1-\sigma} = \beta \left[\frac{1 + \Omega}{\beta - 1} \right]^{(1+\beta)/\beta} \left[\frac{\kappa_D}{\kappa_I} \right]^{1/\beta} \tag{A.23}$$

and

$$\bar{\kappa} = \frac{\beta \kappa_D (1 + \Omega)}{\beta - 1} \tag{A.24}$$

respectively. Notice that expression (A.23) is not exactly the same as that in the corresponding expression (18) in BRN. This constitutes the first correction this paper points at. In what follows I keep track of other expressions that should be rewritten as result of this correction but no major conclusion in BRN is altered by it.

The fact that all savings in the model are directed towards R&D activities and $w = 1$ implies that $S = L_K$. Using this, the consumer's budget constraint and the expression for aggregate operating profits, we find

$$E = \frac{L - L_K}{1 - 1/\sigma} \tag{A.25}$$

This expression shows how aggregate expenditure in final goods is directly linked to the allocation of resources.

Notice that the previous expressions imply that, at steady state, a_D, a_X, $\bar{m}$ and $\bar{\kappa}$ are constant as long as the parameters of the model remain unchanged. Given that the price of knowledge in intensive terms p_K has as sole argument the vector of marginal costs for producing firms $\overrightarrow{a}$, the fact that a_D and $G(a)$ are constant over time at equilibrium implies that p_K is a constant too. In words, once the economy reaches its steady state equilibrium, the endogenous variables determining the distribution of producing and exporting firms, as well as the marginal selling cost of the benchmark firm and the average fixed cost in intensive terms remains unchanged in absence of a shock over some of the parameters of the model.

BRN proposes five different specifications for p_K reflecting the diversity of ways in which it is possible to introduce externalities in the R&D process. The baseline is the *Grossman-Helpman* product-innovation model with knowledge spillovers where productivity in the creation of knowledge increases with knowledge accumulation (n). Allowing for a degree $0 \leq \lambda \leq 1$ of international spillovers the proposed expression for P_K equals $w/(n + \lambda n^*)$ where n^* is the mass of varieties produced in the foreign market and $n^* = n$ by symmetry. This yields $p_K = 1/(1 + \lambda)$ and using (A.24) we reach

$$[p_K\bar{\kappa}]^{GH} = \frac{\beta\kappa_D(1 + \Omega)}{(\beta - 1)(1 + \lambda)} \tag{A.26}$$

A special case of the previous is one where λ depends on trade flows between countries. To fit the *Coe-Helpman* proposal BRN set $\tilde{\lambda} = (a_X/a_D)^k$. Using (A.19) we reach

$$[p_K\bar{\kappa}]^{CH} = \frac{\beta\kappa_D(1 + \Omega)}{(\beta - 1)(1 + \Omega/T)} \tag{A.27}$$

Another possibility is to have *efficiency-linked knowledge spillovers*, which under the assumptions of the model can be introduced setting $p_K = 1/\bar{m}^{1-\sigma}$. This yields

$$[p_K\bar{\kappa}]^{EL} = \left[\frac{(\beta - 1)\kappa_I}{(1 + \Omega)\kappa_D}\right]^{1/\beta} \kappa_D \tag{A.28}$$

A further variation of the canonical model allows for spillovers being the result of *reverse-engineering* learning. Allowing the domestic R&D sector to learn only from varieties that are available at the local market and assuming that the knowledge that can be extracted from local products $\bar{\kappa}$ is lower than that embedded in imported goods $\bar{\kappa}_X = \kappa_I + \kappa_D + \kappa_X$ then $p_K = 1/[\bar{\kappa} + \tilde{\lambda}\bar{\kappa}_X]$ and

$$[p_K\bar{\kappa}]^{RE} = \left[1 + \frac{(\beta - 1)\bar{\kappa}_X\Omega}{(1 + \Omega)\beta\kappa_D T}\right]^{-1} \tag{A.29}$$

Finally, in a *lab-equipment* version of the model knowledge is created using a CES composite of final goods. In our model this sets $p_K = \left(\frac{\sigma-1}{\sigma\bar{m}}\right)^{1-\sigma}$ and

$$[p_K\bar{\kappa}]^{LE} = \left[\frac{\sigma}{\sigma - 1}\right]^{\sigma-1} \left[\frac{(\beta - 1)\kappa_I}{(1 + \Omega)\kappa_D}\right]^{1/\beta} \kappa_D \tag{A.30}$$

Notice that equations (A.28) and (A.30) differ from the corresponding equations (21) and (23) in BRN. This is the outcome of the correction made to the expression for $\bar{m}$ in this paper. These corrections can be considered as minor since no significant results in BRN change as a consequence of them.

3 Dynamics and BGP

Setting L_K as state variable ($\dot{L}_K = 0$), a balanced growth path (BGP) in this economy imposes a constant level of expenditure E (by A.25) and a constant growth rate of varieties g (by A.14). The former yields $r = \rho$ by (A.2), while the latter implies that the discount factor is a constant at BGP. To show this, we can write the value at t of a firm with marginal cost a from its activities in the domestic market as:

$$V(a,t) = \int_t^\infty e^{-(\rho+\delta)(s-t)}\pi(a,s)\mathrm{d}s \underbrace{=}_{(A.11)} \frac{s(a,t)E}{\sigma}\int_t^\infty e^{-(\rho+\delta+g)(s-t)}\mathrm{d}s = \frac{\pi(a,t)}{\rho+\delta+g}$$

Besides selling to the domestic market, a fraction of firms also export to the foreign market. The present value of operating profits from exporting amounts to $\phi V(a,t)$. From the previous expression it is possible to see that the discount rate that firms use is given by

$$\gamma = \rho + \delta + g \tag{A.31}$$

which is constant at BGP.

Inserting (A.31) into the free entry condition in (A.16) we get

$$\frac{E}{\sigma} = p_K\bar{\kappa}(g+\rho+\delta) \tag{A.32}$$

Merging (A.14) and (A.25) gives

$$(1-1/\sigma)E = L - (g+\delta)p_K\bar{\kappa} \tag{A.33}$$

Joining (A.32) and (A.33) it is possible to find the expression for the level of expenditure at BGP:

$$E = L + \rho p_K\bar{\kappa} \tag{A.34}$$

Plugging this expression back into (A.32) gives the constant growth rate of varieties:

$$g = \frac{L}{\sigma p_K\bar{\kappa}} - \frac{\rho(\sigma-1)}{\sigma} - \delta \tag{A.35}$$

It is immediate from this expression that g depends positively on L and negatively on $p_K\bar{\kappa}$. We can also easily show that:

$$\frac{\partial g}{\partial \sigma} = -\frac{1}{\sigma^2}\left[\frac{L}{p_K\bar{\kappa}} + \rho\right] < 0$$

so the growth rate of varieties g depends negatively on σ.

The BGP of this economy is characterized by a constant allocation of labor among sectors. This creates a constant flow of varieties into the economy following

$$n(t) = n(z)e^{g(t-z)} \tag{A.36}$$

The constant entrance of new firms into the economy pushes the ideal price index down at a constant rate. Indeed, using (A.4) and (A.8) we reach:

$$P(t) = \left[\int_0^{\infty} p(m)^{1-\sigma} n(t) \mathrm{d}G(a)\right]^{1/(1-\sigma)} = n(t)^{1/(1-\sigma)} \frac{\bar{m}\sigma}{\sigma - 1} \tag{A.37}$$

which means that the constant rate at which prices and consumption evolve over time at BGP are

$$g_P = \frac{\dot{P}(t)}{P(t)} = \frac{g}{1-\sigma} \quad \text{and} \quad g_C = \frac{[E/\dot{P}(t)]}{E/P(t)} = \frac{g}{\sigma - 1}$$

respectively. Notice that expression (A.37) is correctly stated in footnote 14 but is miswritten in footnote 21 of BRN.

4 Growth effect of openness

A.1 From autarky to free trade

Table A.1 provides results for $p_K\bar{\kappa}$ under autarky ($\Omega = \lambda = 1/T = 0$) and perfect integration ($\Omega = \lambda = T = 1$) for all expressions in (A.26)-(A.30).

Table A.1: Value of $p_K\bar{\kappa}$ in each version of the model

	autarky	full integration
Grossman-Helpman	$\frac{\beta\kappa_D}{\beta-1}$	$\frac{\beta\kappa_D}{\beta-1}$
Coe-Helpman	$\frac{\beta\kappa_D}{\beta-1}$	$\frac{\beta\kappa_D}{\beta-1}$
Efficiency-linked	$\left[\frac{(\beta-1)\kappa_I}{\kappa_D}\right]^{1/\beta} \kappa_D$	$\left[\frac{(\beta-1)\kappa_I}{2\kappa_D}\right]^{1/\beta} \kappa_D$
Reverse-engineering	1	$\left[1 + \frac{(\beta-1)\bar{\kappa}_X}{2\beta\kappa_X}\right]^{1/\beta} \kappa_D$
Lab-Equipment	$\left[\frac{(\beta-1)\kappa_I}{\kappa_D}\right]^{1/\beta} \kappa_D \left[\frac{\sigma}{\sigma-1}\right]^{\sigma-1}$	$\left[\frac{(\beta-1)\kappa_I}{2\kappa_D}\right]^{1/\beta} \kappa_D \left[\frac{\sigma}{\sigma-1}\right]^{\sigma-1}$

A.2 Small change in openness

Table A.2 provides results for the effect of lower trade costs (both sunk costs i.e. $\mathrm{d}T < 0$ or iceberg costs i.e. $\mathrm{d}\phi > 0$) on the product $p_K\bar{\kappa}$.

Table A.2: Effect of lower trade costs on $p_K\bar{\kappa}$

	$\frac{\mathrm{d}p_K\bar{\kappa}}{\mathrm{d}\phi}$	$\frac{\mathrm{d}p_K\bar{\kappa}}{\mathrm{d}T}$
Grossman-Helpman	$\frac{\beta^2\kappa_D\Omega}{(\beta-1)(1+\lambda)\phi}$	$-\frac{\beta\kappa_D\Omega}{(1+\lambda)T}$
Coe-Helpman	$\frac{\beta^2\kappa_D\Omega}{(\beta-1)(1+\Omega/T)^2\phi}\left[1-\frac{1}{T}\right]^*$	$\frac{\beta\kappa_D\Omega\left[(1-\beta)(1+\Omega/T)+\beta\left(\frac{1+\Omega}{T}\right)\right]^{**}}{(\beta-1)(1+\Omega/T)^2T}$
Efficiency-linked	$-\frac{p_K\bar{\kappa}\kappa_D\Omega}{(1+\Omega)\phi}$	$\frac{p_K\bar{\kappa}\kappa_D\Omega(\beta-1)}{(1+\Omega)T\beta}$
Reverse-engineering	$-\left[\frac{p_K\bar{\kappa}}{(1+\Omega)}\right]^2\frac{(\beta-1)\bar{\kappa}_X\Omega}{\kappa_D T\phi}$	$(p_K\bar{\kappa})^2\frac{(\beta-1)\Omega}{\beta(1+\Omega)T}\left[\frac{\bar{\kappa}_X(\beta+\Omega)}{\kappa_D(1+\Omega)T}-1\right]^{***}$
Lab-Equipment	$-\left[\frac{\sigma}{\sigma-1}\right]^{\sigma-1}\frac{p_K\bar{\kappa}\kappa_D\Omega}{(1+\Omega)\phi}$	$\left[\frac{\sigma}{\sigma-1}\right]^{\sigma-1}\frac{p_K\bar{\kappa}\kappa_D\Omega(\beta-1)}{(1+\Omega)\phi\beta}$

* The value is 0 iff $T = 1$

** The sign is + iff $\frac{\beta-1}{\beta} < \frac{(1+\Omega)/T}{(1+\Omega/T)}$

*** The sign is + since $\frac{\bar{\kappa}_X}{\kappa_D T} > 1$ and $\frac{\beta+\Omega}{1+\Omega} > 1$.

Table A.3 summarizes the sign of the effects of lower trade costs on the growth rate g (remember that the effect of trade openness on growth has the opposite sign to the effect on the product $p_K\bar{\kappa}$).

Table A.3: Sign of the effect of lower trade costs on g

	effect on g of	
	$\mathrm{d}\phi > 0$	$\mathrm{d}T < 0$
Grossman-Helpman	$-$	$-$
Coe-Helpman	$-^*$	$+/-^{**}$
Efficiency-linked	$+$	$+$
Reverse-engineering	$+$	$+$
Lab-Equipment	$+$	$+$

* The sign equals 0 iff $T = 1$

** The sign is + iff $\frac{\beta-1}{\beta} < \frac{(1+\Omega)/T}{(1+\Omega/T)}$

*** The sign is + since $\frac{\bar{\kappa}_X}{\kappa_D T} > 1$ and $\frac{\beta+\Omega}{1+\Omega} > 1$.

Inspection of (A.26) reveals that an exogenous increase in the learning parameter ($\mathrm{d}\lambda > 0$) fosters growth in the Grossman-Helpman model.

5 Welfare effect of openness

Welfare at t in this model is a function of the present value of future real consumption. By (A.34), E is constant at BGP so $E(t) = E$. Plugging this and (A.37) into (A.1) yields

$$
\begin{aligned}
U(t) &= \frac{\ln(E)}{\rho} - \int_t^{\infty} e^{-\rho(s-t)} \ln\left(n(s)^{1/(1-\sigma)}\frac{\bar{m}\sigma}{\sigma-1}\right) \mathrm{d}s \\
&= \frac{1}{\rho}\left[\ln E - \ln\left(\frac{\bar{m}\sigma}{\sigma-1}\right)\right] - \int_t^{\infty} e^{-\rho(s-t)} \ln\left(n(s)^{1/(1-\sigma)}\right) \mathrm{d}s \\
&= \frac{1}{\rho}\left[\ln E - \ln\left(\frac{\bar{m}\sigma}{\sigma-1}\right) - \ln\left(n(t)^{1/(1-\sigma)}\right)\right] + \int_t^{\infty} e^{-\rho(s-t)} \frac{gs}{\sigma-1} \mathrm{d}s
\end{aligned}
$$

Then we can write welfare at a given moment $t = 0$ in equilibrium as:

$$
U_0 = \frac{1}{\rho}\left[\ln\left(\frac{E_0}{P_0}\right) + \frac{g}{\rho(\sigma-1)}\right] \tag{A.38}
$$

where $E_0 = E = L + \rho p_K \bar{\kappa}$, $P_0 = n_0^{1/(1-\sigma)} \bar{m}\frac{\sigma}{\sigma-1}$ and g is as in (A.35). This expression is different in many respects from the respective equation (24) in BRN and constitutes the main correction this paper points at in that work as it yields major changes in welfare conclusions. In (A.38), the static welfare effects are reflected in changes in the level of real expenditure at the moment of the change $t = 0$, i.e. $\frac{E_0}{P_0}$, while the dynamic effects are those affecting the growth rate of this level over time $g_C = g/(\sigma - 1)$. Throughout the analysis in this section we set the initial mass of varieties of each equilibrium path at the same level so the comparison focuses in the changes brought about by openness and ignores the evolution of n over time between these two moments. This is equivalent to setting $\mathrm{d}n_0 = 0$ in our comparative exercise.

A generic reduction in trade costs (i.e. $\mathrm{d}\Omega > 0$), which can be the outcome of a reduction in iceberg costs ($\mathrm{d}\phi > 0$) or beachhead costs ($\mathrm{d}T < 0$), has the following effect on welfare:

$$
\frac{\mathrm{d}U_0}{\mathrm{d}\Omega} = \frac{1}{\rho}\left[\underbrace{\frac{\mathrm{d}(p_K\bar{\kappa})}{\mathrm{d}\Omega}\frac{\rho}{E_0}}_{\text{static effect on } E} + \underbrace{\frac{1+\beta}{k(1+\Omega)}}_{\text{static effect on } P} \underbrace{- \frac{\mathrm{d}(p_K\bar{\kappa})}{\mathrm{d}\Omega}\frac{L}{\sigma(\sigma-1)\rho(p_K\bar{\kappa})^2}}_{\text{dynamic effect}}\right]
$$

Finally, when greater openness is the result of an increase in the learning possibilities from foreign production (i.e. $\mathrm{d}\lambda > 0$), the effects are somewhat different. Remember the learning parameter λ is exogenous only in the canonical Grossman-Helpman model. Since λ is absent in (A.37) there is no price effect from $\mathrm{d}\lambda > 0$ so we obtain

$$
\frac{\mathrm{d}U_0}{\mathrm{d}\lambda} = \frac{\mathrm{d}p_K\bar{\kappa}}{\mathrm{d}\lambda}\left[\frac{1}{E} - \frac{L}{(\sigma-1)\sigma(\rho p_K\bar{\kappa})^2}\right] \tag{A.39}
$$

Inspection of (A.26) reveals that $\frac{dp_K\bar{\kappa}}{d\lambda} < 0$ which by (A.38) means that a given increase in learning creates a negative static effect on nominal expenditure and a positive effect on future growth. The total welfare effect is positive if and only if the term in parenthesis is positive.

6 A discussion of welfare effects in a static context

This section shows how welfare results stemming from our model are related to those previously presented in Melitz (2003) and Arkolakis et al. (2012, ACR hereafter).

In those works, there is no room for consumers savings or variety growth so welfare gains from trade are a direct result of reduction in prices. According to ACR, in a Melitz-type model with Pareto distributed firms we should have

$$\mathrm{d}\ln U_0 = -\mathrm{d}\ln P = \frac{\mathrm{d}\ln\Lambda}{\epsilon}$$

were Λ is the share of expenditure in domestic goods and ϵ is the elasticity of imports to changes in international trade costs. We can rewrite our result for the static effect on prices in terms of these two observables to show how it relates to the previous expression and explain the differences. For this let me follow some simple steps.

First, notice that, as explained in ACR, with Pareto-distributed firms $\epsilon = -\alpha$. The share of expenditure in domestic goods is

$$\begin{aligned}\Lambda &= \frac{\int_{\Theta^*} e(\theta)\mathrm{d}\theta}{E} \underbrace{=}_{(A.6)} \left[\frac{\int_{\Theta^*} p(\theta)\mathrm{d}\theta}{P}\right]^{1-\sigma} \underbrace{=}_{(A.37)} \frac{\int_0^\infty a^{1-\sigma} n\mathrm{d}G(a|a_D)}{n\bar{m}^{1-\sigma}} \\ &= \frac{\beta}{\beta-1}\left[\frac{a_D}{\bar{m}}\right]^{1-\sigma} \underbrace{=}_{(A.21)\cap(A.23)} \frac{1}{1+\Omega} \end{aligned} \tag{A.40}$$

where Θ^* is the set of goods produced domestically. This expression shows that when the economy approaches free trade ($\Omega \to 1$) then expenditure in one economy is split in halves between domestic and foreign production ($\Lambda \to 1/2$), while when the economy approaches autarky ($\Omega \to 0$) then consumers can only access domestic products ($\Lambda \to 1$).

Second, let me use (A.37) together with (A.19) and (A.20) to re-write our expression for the price index as follows:

$$P = \frac{\sigma}{\sigma-1}\left[\frac{\beta-1}{\beta n(1+\Omega)}\right]^{\frac{1}{\sigma-1}} a_D \tag{A.41}$$

Third, notice that, by (A.21) we have

$$\frac{\mathrm{d}\ln a_D}{\mathrm{d}\Omega} = -\frac{1}{k(1+\Omega)} = -\frac{\mathrm{d}\ln\Lambda}{\mathrm{d}\Omega}\frac{1}{\epsilon} \tag{A.42}$$

Using (A.41) and (A.42) we can see that the static price effect in our model can be rewritten as

$$\mathrm{d}\ln U_0 = -\mathrm{d}\ln P = \frac{\mathrm{d}\ln\Lambda}{\epsilon}(1+\beta)$$

Comparison of the previous result with that in ACR shows that the static effect on prices is larger in our model. The reason for this is that, in the present model, the mass of varieties (n) is determined by history so a change in trade costs does not immediately affect it. In Melitz (2003), on the other hand, greater openness increases import competition which reduces the mass of existing firms. This effect attenuates the reduction of prices in the Melitz model.

Now, instead of contrasting the static price effect in our model to that in a model with no growth as Melitz, we could impose no growth in our model and compare the resulting price effect to that in Melitz. Setting $g = \rho = 0$ in our model yields endogenous mass of varieties as in Melitz. Indeed, by (A.14), under this setting we obtain

$$n = \frac{L}{\sigma\delta P_K \bar{\kappa}} \underbrace{\Rightarrow}_{(A.41)\cap(A.24)} P = \frac{\sigma}{\sigma - 1}\left[\frac{\sigma\delta\kappa_D P_K}{L}\right]^{\frac{1}{\sigma-1}} a_D \tag{A.43}$$

According to this result, the mass of varieties at equilibrium is a positive function of the size of the economy and the efficiency in the R&D sector and a negative function of the exit rate of firms and the average cost of developing varieties. More importantly, the impact of greater openness over n is determined by how lower trade costs affect P_K and $\bar{\kappa}$. This is not the case in Melitz (2003) where openness cannot yield any efficiency gains in the development of varieties. In fact notice that further imposing $c = 1$ (which yields $P_K = 1$) in the previous result gives us a price equation which exactly matches the inverse of the welfare expression in the original Melitz model (see equation D1 in page 1721 of Melitz 2003). Comparing the expression for prices in (A.43) with one where $P_K = 1$ makes the differences in welfare effects evident. A reduction in variable trade costs in the latter affects only threshold a_D which by (A.42) implies the result in ACR holds. But for the former expression of prices, changes in openness also affect P_K.

In sum, the static price effect in our model is greater than that in Melitz (2003) and ACR since, in the present model, the mass of varieties is determined by history and does not immediately respond to changes in openness. A version of the present model with no growth and savings allows for n to change with openness, but even in that setting our static effect on prices does not quite replicate that in ACR because spillovers in the R&D process affect the cost of entry and therefore the mass of varieties is affected through a channel that is absent in the original Melitz model. We can replicate welfare effects in that model by further setting $P_K = 1$ which implies that there are no spillovers in the knowledge sector and costs are expressed in terms of labour.